

Self-adjointness of a generalized Camassa-Holm equation

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Abstract

It is well known that the Camassa-Holm equation possesses numerous remarkable properties characteristic for KdV type equations. In this paper we show that it shares one more property with the KdV equation. Namely, it is shown in [1, 2] that the KdV and the modified KdV equations are self-adjoint. Starting from the generalization [3] of the Camassa-Holm equation [4], we prove that the Camassa-Holm equation is self-adjoint. This property is important, e.g. for constructing conservation laws associated with symmetries of the equation in question. Accordingly, we construct conservation laws for the generalized Camassa-Holm equation using its symmetries.

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1 Introduction

The Camassa-Holm equation

$$F \equiv u_t - u_{txx} - u u_{xxx} - 2 u_x u_{xx} + 3 u u_x + \kappa u_x = 0 \quad (1)$$

has appeared in [4], [5] as a shallow water wave equation. Here $u(t, x)$ is the fluid velocity in the x direction and κ is an arbitrary constant. Eq. (1) was studied also by Fokas [6] and Fuchsstainer [7].

Clarkson, Mansfield and Priestley [3] studied the third-order nonlinear equation of the form

$$F \equiv u_t - \epsilon u_{txx} - u u_{xxx} - \beta u_x u_{xx} - \alpha u u_x + \kappa u_x = 0 \quad (2)$$

with arbitrary parameters ϵ, α, β . It contains not only the Camassa-Holm equation (1) as a particular case, but also other interesting nonlinear equations such as:

- the Fornberg-Whitham equation [8]

$$u_t - u_{txx} - u u_{xxx} - 3 u_x u_{xx} + u u_x + u_x = 0,$$

- the Rosenau-Hyman equation [9]

$$u_t - u u_{xxx} - 3 u_x u_{xx} - u u_x = 0.$$

Eq. (1), as any evolution type equation, does not have a usual Lagrangian¹. Therefore the classical Noether theorem cannot be employed for constructing conservation laws using symmetries of Eq. (1). On the other hand, a new procedure was developed in [2] for constructing conservation laws associated with symmetries. The new procedure allows one to construct a conservation law using any (Lie point, Lie-Bäcklund, nonlocal, etc.) symmetry of any differential equation. However, the resulting conservation laws involve, in general, not only the solutions of the original equation, but also so-called *nonlocal variables*, namely solutions of the adjoint equation. The nonlocal variables can be eliminated if the equation under consideration is quasi self-adjoint (or, in particular, self-adjoint) in the sense defined in [11]. Therefore the quasi self-adjointness is important for constructing conservation laws. Accordingly, we start our paper with investigating the quasi self-adjointness of the generalized Camassa-Holm equation (2). Our construction require the concepts of the *formal Lagrangian* and the *adjoint equation* for Eq. (2).

2 Formal Lagrangian and adjoint equation

According to the procedure suggested in [2], we introduce the *formal Lagrangian*

$$\mathcal{L} \equiv vF = v [u_t - \epsilon u_{txx} - u u_{xxx} - \beta u_x u_{xx} - \alpha u u_x + \kappa u_x] \quad (3)$$

and define the adjoint equation $F^* = 0$ by

$$F^* \equiv \frac{\delta \mathcal{L}}{\delta u} = 0. \quad (4)$$

¹By this we mean that there is no function $L(x, t, u, u_x, u_t, u_{xx}, u_{tx}, \dots)$ such that Eq. (1) is identical with the Euler-Lagrange equation

$$\frac{\delta L}{\delta u} \equiv \frac{\partial L}{\partial u} - D_x \frac{\partial L}{\partial u_x} - D_t \frac{\partial L}{\partial u_t} + D_x^2 \frac{\partial L}{\partial u_{xx}} + D_t D_x \frac{\partial L}{\partial u_{tx}} + \dots = 0.$$

However it is shown in [10] that the CH equation can be written as the Euler-Poincaré equation. It also admits two canonical Hamiltonian representations, namely in terms of the *Clebsch variables* and the so-called *peakon variables*, obtained by using different momentum maps.

Here $v = v(t, x)$ is a new dependent variable,

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - D_i D_j D_k \frac{\partial}{\partial u_{ijk}} + \dots \quad (i, j, k = 1, 2),$$

is the variational derivative and

$$D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + v_i \frac{\partial}{\partial v} + u_{ij} \frac{\partial}{\partial u_j} + v_{ij} \frac{\partial}{\partial v_j} + u_{ijk} \frac{\partial}{\partial u_{jk}} + v_{ijk} \frac{\partial}{\partial v_{jk}} + \dots$$

is the operator of total differentiation with respect to x^i ($x^1 = t, x^2 = x$). The usual convention of summation over repeated indices is used.

We have

$$F^* \equiv \frac{\partial \mathcal{L}}{\partial u} - D_t \frac{\partial \mathcal{L}}{\partial u_t} - D_x \frac{\partial \mathcal{L}}{\partial u_x} + D_x^2 \frac{\partial \mathcal{L}}{\partial u_{xx}} - D_x^3 \frac{\partial \mathcal{L}}{\partial u_{xxx}} - D_t D_x^2 \frac{\partial \mathcal{L}}{\partial u_{txx}} = 0 \quad (5)$$

and, taking into account the relation (3), we obtain

$$\begin{aligned} F^* = & -vu_{xxx} - \alpha vu_x - D_t(v) - D_x(-\beta vu_{xx} - \alpha vu + \kappa v) \\ & + D_x^2(-\beta vu_x) - D_x^3(-vu) + \epsilon D_t D_x^2(v). \end{aligned}$$

Performing here the differentiations, we arrive at the following adjoint Eq. (4):

$$F^* \equiv -v_t + \epsilon v_{txx} + uv_{xxx} + (3 - \beta)(u_x v_{xx} - v_x u_{xx}) + \alpha uv_x - \kappa v_x = 0. \quad (6)$$

Definition 1. An equation $F = 0$ is said to be *quasi self-adjoint* [11] if there exists a function

$$v = \varphi(u), \quad \varphi'(u) \neq 0, \quad (7)$$

such that

$$F^*|_{v=\varphi(u)} = \lambda F \quad (8)$$

with an undetermined coefficient λ . If in (7) $\varphi(u) = u$, we say that the equation $F = 0$ is *self-adjoint*.

Taking into account the expression (6) of F^* and using Eq. (7) together with its consequences

$$\begin{aligned} v_t &= \varphi' u_t, \quad v_x = \varphi' u_x, \\ v_{tx} &= \varphi' u_{tx} + \varphi'' u_t u_x, \quad v_{xx} = \varphi' u_{xx} + \varphi'' u_x^2, \\ v_{txx} &= \varphi' u_{txx} + 2\varphi'' u_x u_{tx} + \varphi'' u_t u_{xx} + \varphi''' u_t u_x^2, \\ v_{xxx} &= \varphi' u_{xxx} + 3\varphi'' u_x u_{xx} + \varphi''' u_x^3, \end{aligned}$$

we rewrite Eq. (8) in the following form:

$$\begin{aligned} & -\varphi' u_t + \alpha \varphi' u u_x - \kappa \varphi' u_x + [\varphi''' u - (\beta - 3) \varphi''] u_x^3 + \varphi' u u_{xxx} \\ & + \epsilon \varphi' u_{txx} + [3 \varphi'' u - 2(\beta - 3) \varphi'] u_x u_{xx} + 2 \epsilon \varphi'' u_x u_{tx} + \epsilon \varphi''' u_t u_x^2 \\ & + \epsilon \varphi'' u_t u_{xx} = \lambda (u_t - \epsilon u_{txx} - u u_{xxx} - \alpha u u_x - \beta u_x u_{xx} + \kappa u_x). \end{aligned} \quad (9)$$

Eq. (9) should be satisfied identically in all variables u_t, u_x, u_{xx}, \dots . Comparing the coefficients of u_t in both sides of Eq. (9), we obtain $\lambda = -\varphi'$. Then we equate the coefficients of $u_x u_{tx}$ and get:

$$\epsilon \varphi'' = 0. \quad (10)$$

According to Eq. (10), the procedure splits into two cases:

$$\epsilon = 0, \quad (11)$$

$$\epsilon \neq 0, \quad \varphi'' = 0. \quad (12)$$

In the case (11) we compare the coefficients of $u_x u_{xx}$ and arrive at the equation

$$\varphi'' u + (\beta - 2) \varphi' = 0. \quad (13)$$

Integrating Eq. (13) we obtain

$$\varphi(u) = \begin{cases} a + b u^{\beta-1}, & \beta \neq 1, \\ a + b \ln u, & \beta = 1, \end{cases} \quad (14)$$

where a and b are arbitrary constants. The coefficients of the other terms in both sides of Eq. (9) are equal due to Eq. (13). Hence we have proved that the equation

$$u_t - u u_{xxx} - \beta u_x u_{xx} - \alpha u u_x + \kappa u_x = 0 \quad (15)$$

is quasi self-adjoint and that the substitution (7) has the form

$$v = \begin{cases} a + b u^{\beta-1}, & \beta \neq 1, \\ a + b \ln u, & \beta = 1. \end{cases} \quad (16)$$

Now we consider the case (12). In this case the comparison of the coefficients for $u_x u_{xx}$ yields that $\beta = 2$. The coefficients of the other terms in both sides of Eq. (9) are equal. Thus we have proved that the equation

$$u_t - \epsilon u_{txx} - u u_{xxx} - 2 u_x u_{xx} - \alpha u u_x + \kappa u_x = 0 \quad (17)$$

is quasi self-adjoint for any parameters ϵ, α, κ and that the substitution (7) has the following form:

$$v = a + b u. \quad (18)$$

We can take, in particular, $a = 0$ and $b = 1$. Hence Eq. (17) coincides with its adjoint equation after the substitution $v = u$. According to [1], it means that the *generalized Camassa-Holm equation* (17) is self-adjoint.

In conclusion of this section we note that the Fornberg-Whitham equation

$$u_t - u_{txx} - u u_{xxx} - 3 u_x u_{xx} + u u_x + u_x = 0 \quad (19)$$

is not quasi self-adjoint in the sense of Definition 1.

3 Conservation laws

3.1 General form

We rewrite the formal Lagrangian (3) in the symmetric form

$$\mathcal{L} = v \left[u_t - \frac{\epsilon}{3} (u_{txx} + u_{xtx} + u_{xxt}) - u u_{xxx} - \alpha u_x u_{xx} - \beta u u_x + \kappa u_x \right]. \quad (20)$$

Eq. (2) is said to have a *nonlocal conservation law* if there exists a vector $\mathbf{C} = (C^1, C^2)$ satisfying the equation

$$D_t(C^1) + D_x(C^2) = 0 \quad (21)$$

on any solution of the system (2), (6). Eq. (2) has a *local conservation law* if (21) is satisfied on any solution of Eq. (2).

The conserved vector corresponding to an operator

$$X = \xi^1(t, x, u) \frac{\partial}{\partial t} + \xi^2(t, x, u) \frac{\partial}{\partial x} + \eta(t, x, u) \frac{\partial}{\partial u} \quad (22)$$

admitted by Eq. (2) is obtained by the following formula [2]:

$$\begin{aligned} C^i = \xi^i \mathcal{L} + W \left[\frac{\partial \mathcal{L}}{\partial u_i} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) \right] \\ + D_j(W) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}} \right) \right] + D_j D_k(W) \frac{\partial \mathcal{L}}{\partial u_{ijk}}, \end{aligned} \quad (23)$$

where $i, j, k = 1, 2$ and

$$W = \eta - \xi^i u_i.$$

We will construct the conserved vectors (23) using the Lie point symmetries (22) of Eq. (2) found in [3].

3.2 Rosenau-Hyman equation

Letting in Eq. (15) $\alpha = 1$, $\beta = 3$, $\kappa = 0$, we obtain the Rosenau-Hyman equation

$$u_t - u u_{xxx} - 3 u_x u_{xx} - u u_x = 0. \quad (24)$$

In this case, the formal Lagrangian (20) and the substitution (14) assume the forms

$$\mathcal{L} = v [u_t - u u_{xxx} - 3 u_x u_{xx} - u u_x] \quad (25)$$

and

$$v = a + b u^2, \quad (26)$$

respectively. We construct the conservation law associated with the scaling symmetry

$$X = u \frac{\partial}{\partial u} - t \frac{\partial}{\partial t}. \quad (27)$$

For this symmetry we have $W = u + t u_t$. Writing the quantities (23) without the term $\xi^i \mathcal{L}$ since the Lagrangian \mathcal{L} is equal to zero on solutions of Eq. (24) and taking into account the structure of the formal Lagrangian (25), we obtain

$$C^1 = W \frac{\partial \mathcal{L}}{\partial u_t}, \quad (28)$$

$$\begin{aligned} C^2 = W & \left[\frac{\partial \mathcal{L}}{\partial u_x} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xx}} \right) + D_x^2 \left(\frac{\partial \mathcal{L}}{\partial u_{xxx}} \right) \right] \\ & + D_x(W) \left[\frac{\partial \mathcal{L}}{\partial u_{xx}} - D_x \left(\frac{\partial \mathcal{L}}{\partial u_{xxx}} \right) \right] + D_x^2(W) \frac{\partial \mathcal{L}}{\partial u_{xxx}}. \end{aligned} \quad (29)$$

Substituting in (28) and (29) the expression (25) for \mathcal{L} , we get

$$C^1 = v W, \quad (30)$$

$$\begin{aligned} C^2 = & (-u v + u_x v_x - v u_{xx} - u v_{xx}) W \\ & + (u v_x - 2 v u_x) D_x(W) - u v D_x^2(W). \end{aligned} \quad (31)$$

Now we substitute in Eq. (30) the expression $W = u + t u_t$, eliminate u_t by using Eq. (24) and obtain:

$$\begin{aligned} C^1 = & u v + t v (u u_{xxx} + u u_x + 3 u_x u_{xx}) \\ = & u v + t u v u_x - \frac{3}{2} t v_x u_x^2 - t D_x(u v) u_{xx} \\ & + D_x \left(t u v u_{xx} + \frac{3}{2} t v u_x^2 \right). \end{aligned} \quad (32)$$

We can shift the last term in Eq. (32) into C^2 by using the identity

$$D_t(\tilde{C}^1 + D_x(A)) + D_x(C^2) = D_t(\tilde{C}^1) + D_x(C^2 + D_t(A))$$

and obtain

$$C^1 = uv + tuv u_x - \frac{3}{2} t v_x u_x^2 - t D_x(uv) u_{xx}. \quad (33)$$

Now we substitute in Eq. (33) the expression (26) for v , shift the terms of the form $D_x(\dots)$ into C^2 and finally arrive at the conserved vector with the following components:

$$\begin{aligned} C^1 &= au + bu^3, \\ C^2 &= -a \left(\frac{1}{2} u^2 + u_x^2 + u u_{xx} \right) - b \left(\frac{3}{4} u^4 + 3 u^3 u_{xx} \right). \end{aligned} \quad (34)$$

The vector (34) is a linear combination with constant coefficients a and b of the following two linearly independent conserved vectors:

$$C^1 = u, \quad C^2 = -\frac{1}{2} u^2 - u_x^2 - u u_{xx} \quad (35)$$

and

$$C^1 = u^3, \quad C^2 = -\frac{3}{4} u^4 - 3 u^3 u_{xx}. \quad (36)$$

The conservation equation (21) for the vector (35) coincides with Eq. (24), whereas the vector (36) provides a new conservation law for the Rosenau-Hyman equation.

3.3 Camassa-Holm equation

For the Camassa-Holm equation (1) the formal Lagrangian (20) is written

$$\mathcal{L} = v \left[u_t - \frac{1}{3} (u_{txx} + u_{xtx} + u_{xxt}) - uu_{xxx} - 2u_x u_{xx} + 3uu_x + \kappa u_x \right]. \quad (37)$$

We will construct the conservation law by taking the substitution (18) of the particular form $v = u$. We use the following symmetry of Eq. (1):

$$X = -2t \frac{\partial}{\partial t} + \kappa t \frac{\partial}{\partial x} + (\kappa + 2u) \frac{\partial}{\partial u}. \quad (38)$$

Proceeding as in Section 3.2, we obtain the following conserved vector associated with the symmetry (38):

$$\begin{aligned} C^1 &= 2(u^2 + u_x^2) + \kappa u, \\ C^2 &= 4(u^3 - u^2 u_{xx} - u u_{tx}) + \kappa \left(\frac{7}{2} u^2 - \frac{1}{2} u_x^2 - u u_{xx} - u_{tx} + \kappa u \right). \end{aligned} \quad (39)$$

When $\kappa = 0$ in Eq. (1) the symmetry (38) takes the form (27), and the conserved vector (39) becomes

$$C^1 = u^2 + u_x^2, \quad C^2 = 2(u^3 - u^2 u_{xx} - u u_{tx}). \quad (40)$$

It is shown in [1] that the well known infinite series of conservation laws of the KdV equation can be obtained by applying the formulae (23) to the infinite set of Lie-Bäcklund and non-local symmetries of the KdV equation. The similar procedure can be applied to the Camassa-Holm and Rosenau-Hyman equations. This is a topic for further research.

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